

# Pure Subnormal Operators Have Cyclic Adjoints

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We shall prove that a pure subnormal operator has a cyclic adjoint. This answers a question raised by J. Deddens and W. Wogen in 1976. We first prove that a subnormal operator  $S$  on a Hilbert space  $\mathcal{H}$  has a cyclic adjoint if and only if there exists a compactly supported Borel measure  $\mu$  in the complex plane and a one-to-one linear map  $A: \mathcal{H} \rightarrow L^2(\mu)$  such that  $AS = N_{\mu}A$  where  $N_{\mu} = M_{\bar{z}}$  on  $L^2(\mu)$ .

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is also used to give new proofs of some known results on the cyclicity of adjoints.

An application of the main result shows that every pure subnormal operator has a matrix representation that is “almost” lower triangular. © 1999 Academic Press

**Key Words:** cyclic adjoint; intertwining maps; subnormal operators.

## 1. INTRODUCTION

In 1950 P. Halmos [11] introduced the class of subnormal operators. As early as 1955, J. Bram [3] characterized the cyclic normal operators and used this to prove that if  $S = M_z$  on  $\mathcal{H}$ , where  $\mathcal{H}$  is a closed  $z$  invariant subspace of  $L^2(\mu)$ , then  $S$  has a cyclic adjoint (also see Conway [8], p. 234). This gave many non-trivial examples of subnormal operators with cyclic adjoints and paved the way for a natural question.

In 1976 J. Deddens and W. Wogen asked (see Conway [8] p. 234 or Wogen [18]): which subnormal operators have a cyclic adjoint? They also asked the same question for hyponormal operators. A “test question” was also raised; namely, if  $S$  is the dual of the Bergman operator (so  $S = M_{\bar{z}}$  on  $L_a^2(\mathbb{D})^\perp$ ), then does  $S \oplus S$  have a cyclic adjoint? Although progress has been made on the general problem, even this test question remained unsettled; until now.

In this paper we shall prove that every pure subnormal operator has a cyclic adjoint. We shall also show that if a subnormal operator is not pure, then it has a cyclic adjoint if and only if its normal part is cyclic. Our technique also allows us to give new proofs of some classical results on cyclicity.

An application of the main result implies that every pure subnormal operator has a matrix representation that is “almost” lower triangular; in the sense that the only non-zero entries above the main diagonal are on the super diagonal. This result is the best possible for one can easily find subnormal operators that do not have a lower triangular matrix representation.

Over the years, there have been several results suggesting that every pure subnormal operator may have a cyclic adjoint. D. Sarason was the first to prove that pure isometries have cyclic adjoints; his solution is in Halmos’ problem book (see Halmos [13] #160).

In 1978 W. Wogen [18] proved an amazing result, namely that the collection of adjoints of (nonscalar) multiplication operators on  $H^2(\mathbb{D})$  has a common cyclic vector. Thus if  $f \in H^\infty(\mathbb{D})$  is nonconstant, not only is  $M_f^*$  cyclic on  $H^2(\mathbb{D})$ , but there is one function in  $H^2(\mathbb{D})$  that is a cyclic vector for any operator in the set  $\{M_\phi^* : \phi \in H^\infty(\mathbb{D}) \text{ and } \phi \text{ is not constant}\}$ . In fact, he showed that the set of common cyclic vectors is dense in  $H^2(\mathbb{D})$ .

Wogen [18] also proved that pure quasinormal operators have cyclic adjoints by proving a more general result about operators with a triangular matrix representation.

In 1978, Clancey and Rogers [6] proved that if  $T$  is any operator such that  $\{\ker(T - \lambda)^* : \lambda \in \sigma(T) - \sigma_{ap}(T)\}$  has dense linear span, then  $T^*$  has a dense set of cyclic vectors. In particular, this implies that a pure hyponormal operator whose approximate point spectrum has area zero will have a cyclic adjoint.

In 1988, K. Chan [5] extended Wogen’s result on common cyclic vectors to Hilbert spaces  $\mathcal{H}$  of analytic functions satisfying certain reasonable criteria; leaving out, however, some natural spaces.

In 1990, P. Bourdon and J. Shapiro [4] extended Wogen’s result on multipliers of  $H^2(\mathbb{D})$  to multipliers of *any* Hilbert space of functions analytic in a domain in  $\mathbb{C}^n$ ; thus removing the extra assumptions of Chan. Both Bourdon and Shapiro, and Clancey and Rogers, use the ideas of spectral synthesis in their proofs.

In this paper we shall introduce a simple method of comparing operators; namely, if  $S$  acts on  $\mathcal{H}$  and  $T$  acts on  $\mathcal{K}$ , then we shall say that  $S \succ T$  if there exists a one-to-one linear map  $A: \mathcal{H} \rightarrow \mathcal{K}$  such that  $AS = TA$ . One easily sees that if  $S \succ T$  and  $T^*$  is cyclic, then  $S^*$  is cyclic.

In Section 2 we shall use this comparison method to prove a (stronger) version of the result of Clancey and Rogers [6]. Also the comparison method allows us to reduce the several variable result of Bourdon and Shapiro [4] to the one variable result of Wogen [18].

In Section 3 we will use the comparison method to get a necessary and sufficient condition for a subnormal operator to have a cyclic adjoint; namely, if  $S$  is subnormal, then  $S^*$  is cyclic if and only if there exists a

cyclic normal operator  $N$  such that  $S \succ N$ . In proving this, a new form of cyclicity is introduced.

We say that  $S$  is *strongly \*-cyclic* if there exists a (strong \*-cyclic) vector  $v$  such that  $\{S^{*n}S^k v; n, k \geq 0\}$  has dense linear span (notice that the adjoints are always on the left). Clearly, an operator with a cyclic adjoint is strongly \*-cyclic; and a strongly \*-cyclic operator is \*-cyclic. Here it is also shown that a subnormal operator has a cyclic adjoint if and only if it is strongly \*-cyclic.

The proof that for every pure subnormal operator  $S$ , there exists a one-to-one map intertwining  $S$  and a cyclic normal operator consists of two steps.

First it is shown that if  $S_n = M_z$  on  $\mathcal{H}_n \subseteq L^2(\mu_n)$  is a bounded sequence of pure subnormal operators, then  $(\bigoplus_n S_n) \succ N$  for some cyclic normal operator  $N$ .

The second step involves showing that if  $S$  is any pure subnormal operator, then  $S \succ (\bigoplus_n S_n)$  where the  $S_n$ 's have the above form.

In Section 4 a few additional results are obtained. For instance, here we discuss subnormal operators that are not pure. It is shown that a subnormal operator has a cyclic adjoint precisely when its normal part is cyclic.

We also characterize the strong \*-cyclic vectors for subnormal operators in terms of some natural intertwining maps. For example, if  $S$  is the unilateral shift, then the cyclic vectors for  $S^*$  may be characterized in terms of pseudocontinuations (see [10]); however, we shall see that every non-zero vector is a strong \*-cyclic vector for  $S$ .

Also in this section it is proven that every pure hyponormal operator is \*-cyclic.

## 2. GENERAL OPERATORS

In this section we shall state and prove the basic comparison method and use it to give new proofs of the results of Bourdon and Shapiro [4] and Clancey and Rogers [6].

The following result is the basic comparison method used throughout the paper. The result is elementary and certainly well known.

**PROPOSITION 2.1** (Comparison method). *Suppose  $S: \mathcal{H} \rightarrow \mathcal{H}$  and  $T: \mathcal{K} \rightarrow \mathcal{K}$  are bounded linear operators on Hilbert spaces. If there exists a one-to-one bounded linear operator  $A: \mathcal{H} \rightarrow \mathcal{K}$  such that  $AS = TA$  and  $T^*$  is cyclic, then  $S^*$  is cyclic.*

*In fact, if  $v$  is a cyclic vector for  $T^*$ , then  $A^*v$  is a cyclic vector for  $S^*$ .*

*Proof.* Since  $AS = TA$  it follows that  $S^*A^* = A^*T^*$ . If  $p(z)$  is any (analytic) polynomial, then  $p(S^*)A^* = A^*p(T^*)$ . Now let  $v$  be a cyclic

vector for  $T^*$ , so  $p(S^*) A^*v = A^*p(T^*)v$  holds. Since  $A$  is one-to-one,  $A^*$  has dense range, thus as  $p$  varies it follows that  $A^*v$  is cyclic for  $S^*$ . ■

*Remark.* In Proposition 2.1 one may replace the word cyclic by hypercyclic or supercyclic and the comparison still holds. Furthermore, one may also use this in a Banach space setting where one considers operators whose adjoints are cyclic in the weak\* topology. Note also that if  $T^*$  has a dense set of cyclic vectors, then  $S^*$  also has a dense set of cyclic vectors.

For convenience we introduce a partial ordering on operators by saying that  $S \succ T$  if  $S \in \mathcal{B}(\mathcal{H})$  and there exists a one-to-one bounded linear operator  $A: \mathcal{H} \rightarrow \mathcal{K}$  such that  $AS = TA$ .

Thus Proposition 2.1 may be stated as follows: if  $S \succ T$  and  $T^*$  is cyclic, then  $S^*$  is also cyclic.

**COROLLARY 2.2** *If  $T$  is an extension of  $S$  and  $T^*$  is cyclic, then  $S^*$  is cyclic.*

*Proof.* If  $S$  acts on  $\mathcal{H}$  and  $T$  acts on  $\mathcal{K}$ , then let  $A: \mathcal{H} \rightarrow \mathcal{K}$  be the inclusion map. It follows that  $A$  is one-to-one and intertwines  $S$  and  $T$ , so the result follows. ■

Recall that a normal operator is \*-cyclic if and only if it is unitarily equivalent to an operator of the form  $M_z$  on  $L^2(\mu)$  for some compactly supported Borel measure  $\mu$  in the complex plane. The following result of Bram [3] (also see Conway [8], p. 232) shows that \*-cyclic normal operators are actually cyclic.

**THEOREM 2.3.** *If  $\mu$  is a compactly supported Borel measure in the complex plane and  $N_\mu$  is multiplication by  $z$  on  $L^2(\mu)$ , then  $N_\mu$  has a cyclic vector in  $L^\infty(\mu)$ .*

*Furthermore, the set of cyclic vectors for  $N_\mu$  is dense in  $L^2(\mu)$ .*

*Proof.* The fact that  $N_\mu$  is cyclic and has a bounded cyclic vector may be found in Bram [3] (also see Conway [8], p. 232). To see that the cyclic vectors are dense, let  $\phi$  be a cyclic vector for  $N_\mu$ . Thus  $|\phi| > 0$   $\mu$ -a.e.. Now consider the set  $\{\phi\psi: \psi \in L^\infty(\mu) \text{ and } |\psi| > 0 \text{ a.e.}\}$ . One may easily see that this set consists of cyclic vectors for  $N_\mu$  and is dense in  $L^2(\mu)$ . ■

*Remark.* If  $f$  is a bounded one-to-one function defined  $\mu$ -a.e., then multiplication by  $f$  on  $L^2(\mu)$  is unitarily equivalent to multiplication by  $z$  on  $L^2(v)$  where  $v = \mu \circ f^{-1}$ . Hence  $M_f$  on  $L^2(\mu)$  is cyclic and has a dense set of cyclic vectors.

**COROLLARY 2.4.** *If  $S$  is a bounded linear operator on a Hilbert space and  $S \succ N$  for some cyclic normal operator  $N$ , then  $S^*$  has a dense set of cyclic vectors.*

In [18] Wogen proved that there exists a common cyclic vector for the collection of adjoints of (nonscalar) multiplication operators on the Hardy space  $H^2(\mathbb{D})$ . In [4] Bourdon and Shapiro extended Wogen's result to any Hilbert space of analytic functions. We now show that the comparison method allows us to obtain this more general result rather easily.

**THEOREM 2.5.** *If  $\mathcal{H}$  is a Hilbert space of analytic functions on a domain  $G$  in  $\mathbb{C}^n$  and  $f$  is any nonconstant multiplier of the space  $\mathcal{H}$ , then  $M_f$  has a cyclic adjoint on  $\mathcal{H}$ . Furthermore, there is a dense set of common cyclic vectors for the set of adjoints of nonscalar multiplication operators on  $\mathcal{H}$ .*

The next Lemma allows us to reduce the several variable problem to the one variable case. The author would like to thank Fedja Nazarov for the following proof. Notice that the Lemma is trivial when  $n = 1$ .

**LEMMA 2.6.** *If  $G$  is a domain in  $\mathbb{C}^n$ , then there exists a bounded analytic function  $\Phi: \mathbb{D} \rightarrow G$  whose range is compactly contained in  $G$  and such that the composition operator  $C_\Phi: \text{Hol}(G) \rightarrow \text{Hol}(\mathbb{D})$  is one-to-one.*

*Proof.* It suffices to find an analytic function  $\Phi: \mathbb{D} \rightarrow \Delta$  where  $\Delta$  is a polydisk compactly contained in  $G$  such that  $\text{cl}[\Phi(\mathbb{D})]$  has nonempty interior. By translating and scaling we may assume that  $\Delta = \mathbb{D}^n$ , the unit polydisk.

Let  $\{z_k\} \subseteq \mathbb{D}$  be an interpolating sequence for  $H^\infty(\mathbb{D})$ . So there exists an  $M > 0$  such that if  $\{a_k\}$  is any bounded sequence of complex numbers, then there exists an  $\phi \in H^\infty(\mathbb{D})$  such that  $\phi(z_k) = a_k$  and  $\|\phi\|_\infty \leq M \|\{a_k\}\|_\infty$ .

Let  $\{\mathbf{a}_k = (a_{k,1}, \dots, a_{k,n})\}$  be a countable dense subset of the polydisk of radius  $1/M$ . Now for each  $j \in \{1, \dots, n\}$  there exists a function  $\phi_j \in H^\infty(\mathbb{D})$  such that  $\phi_j(z_k) = a_{k,j}$  and  $\|\phi_j\|_\infty \leq M \|\{a_{k,j}\}\|_\infty = 1$ . Thus if we let  $\Phi = (\phi_1, \dots, \phi_n)$ , then  $\Phi$  is an analytic function mapping the unit disk  $\mathbb{D}$  into  $\mathbb{D}^n$  such that  $\Phi(z_k) = \mathbf{a}_k$ . Hence  $\text{cl}[\Phi(\mathbb{D})]$  has nonempty interior. ■

*Proof of Theorem 2.5.* First suppose that  $n = 1$ . Since  $f$  is non-constant there exists a point  $z_0 \in G$  such that  $f'(z_0) \neq 0$ . Now choose a disk  $\Delta$  centered at  $z_0$  and compactly contained in  $G$  such that  $f$  is univalent on  $\Delta$ . Also let  $\mu$  be any positive finite Borel measure on  $\Delta$ , except a finite sum of point masses. If  $A: \mathcal{H} \rightarrow L^2(\mu)$  is the restriction map, then  $A$  is one-to-one and intertwines  $(M_f, \mathcal{H})$  and  $(M_f, L^2(\mu))$ . Since  $f$  is one-to-one on  $\Delta$ ,  $M_f$  is cyclic on  $L^2(\mu)$ . Hence Corollary 2.4 applies to say that  $(M_f^*, \mathcal{H})$  has a dense set of cyclic vectors.

Now suppose that  $n > 1$ . Let  $\Phi: \mathbb{D} \rightarrow G$  be the function guaranteed by the Lemma. Thus, the composition operator  $C_\Phi: \mathcal{H} \rightarrow H^2(\mathbb{D})$  is a well defined one-to-one operator. If  $f$  is a multiplier of  $\mathcal{H}$ , then  $C_\Phi$  intertwines multiplication by  $f$  on  $\mathcal{H}$  with multiplication by  $f \circ \Phi$  on  $H^2(\mathbb{D})$ . By the above case ( $n = 1$ ), multiplication by  $f \circ \Phi$  has a cyclic adjoint on  $H^2(\mathbb{D})$ ; it then follows by Proposition 2.1 that multiplication by  $f$  has a cyclic adjoint on  $\mathcal{H}$ .

For the common cyclic vectors, since W. Wogen [18] proved that there is a dense set of common cyclic vectors for the adjoints of nonconstant multiplication operators on  $H^2(\mathbb{D})$ . It follows by Proposition 2.1 that  $C_\Phi^*$  maps common cyclic vectors in  $H^2(\mathbb{D})$  to common cyclic vectors in  $\mathcal{H}$ . Furthermore, since  $C_\Phi$  is one-to-one,  $C_\Phi^*$  has dense range, thus the set of common cyclic vectors in  $\mathcal{H}$  is dense. ■

*Remark.* In the previous result it was not necessary to assume that  $\mathcal{H}$  contains the polynomials or has any division properties. It is even possible for the functions in  $\mathcal{H}$  to have a common zero in  $G$ . Thus this result applies, for instance, to invariant subspaces of the Bergman space.

Notice that the same technique also applies in a Banach space setting. Thus the adjoints of nonscalar multiplication operators on a Banach space of analytic functions have a common weak\* cyclic vector.

Next we turn our attention to a result of Clancey and Rogers [6]. They proved that if  $T$  is any operator such that  $\{ker(T - \lambda)^*: \lambda \in \sigma(T) - \sigma_{ap}(T)\}$  has dense linear span, then  $T^*$  has a dense set of cyclic vectors. Their proof used ideas from spectral synthesis. We now prove it by showing that  $T \succ N$  for some cyclic normal operator  $N$ , and then appeal to Corollary 2.4.

**THEOREM 2.7.** *If  $T \in \mathcal{B}(\mathcal{H})$  and  $\{ker(T - \lambda)^*: \lambda \in \sigma(T) - \sigma_{ap}(T)\}$  has dense linear span, then there exists open disks  $\Delta_n \subseteq \sigma(T) - \sigma_{ap}(T)$  and a one-to-one linear operator  $A: \mathcal{H} \rightarrow \bigoplus_n H^2(\Delta_n)$  such that  $AT = SA$  where  $S = \bigoplus_n M_z$  on  $\bigoplus_n H^2(\Delta_n)$ .*

*In particular,  $T^*$  has a dense set of cyclic vectors.*

Before proving this we need a few preliminary results.

**PROPOSITION 2.8.** *If  $\{\Delta_n\}$  is any bounded collection of open disks and  $S_n = M_z$  on  $H^2(\Delta_n)$ , then  $(\bigoplus_n S_n^*)$  has a dense set of cyclic vectors.*

*Proof.* By shrinking the disks, we may choose disks  $\Delta'_n \subseteq \Delta_n$  such that  $\Delta'_n \neq \Delta'_k$  when  $n \neq k$ . Now let  $A: \bigoplus H^2(\Delta_n) \rightarrow \bigoplus H^2(\Delta'_n)$  be the restriction map. Notice that  $A$  is well defined and one-to-one. Also  $\bigoplus H^2(\Delta'_n)$  is a pure subspace of  $L^2(\mu)$  where  $\mu = \sum_n (1/2^n) \mu_n$  and  $\mu_n$  is arc length measure on the boundary of  $\Delta'_n$ . This follows because the measures  $\{\mu_n\}$  are pairwise singular. If  $S = \bigoplus_n M_z$  on  $\bigoplus H^2(\Delta'_n)$ , then  $A$  intertwines

$(\bigoplus_n S_n)$  and  $S$ . Thus  $(\bigoplus_n S_n) \succ S \succ N_\mu$ . So  $(\bigoplus_n S_n^*)$  has a dense set of cyclic vectors, by Corollary 2.4. ■

The next proposition says that if  $T^*$  has a large supply of eigenvectors, then we can always find coanalytic cross sections of a certain vector bundle. This proposition is well known (see Conway [8], p. 64) and actually a special case of a more general result due to Cowen and Douglas [9]. We shall sketch an elementary proof below.

**PROPOSITION 2.9.** *If  $T \in \mathcal{B}(\mathcal{H})$ ,  $\lambda \in \sigma(T) - \sigma_{ap}(T)$  and  $h_0$  is a nonzero vector in  $\ker(T - \lambda)^*$ , then there is an open disk  $\Delta$  centered at  $\lambda$  and a function  $h: \Delta \rightarrow \mathcal{H}$  such that:*

1.  $h(\lambda) = h_0$ ;
2.  $h(z) \neq 0$  and  $h(z) \in \ker(T - z)^*$  for all  $z \in \Delta$ ;
3. for each  $g \in \mathcal{H}$ , the function  $z \rightarrow \langle g, h(z) \rangle$  is analytic on  $\Delta$ .

*Proof.* We may assume  $\lambda = 0$ . Thus since  $0 \in \sigma(T) - \sigma_{ap}(T)$ ,  $T$  is left invertible. So let  $B \in \mathcal{B}(\mathcal{H})$  be such that  $BT = I$ . Now let  $\Delta = \{z: |z| < \|B\|^{-1}\}$  and define  $h: \Delta \rightarrow \mathcal{H}$  by  $h(z) = (I - \bar{z}B^*)^{-1} h_0$ . Clearly,  $h(0) = h_0$  and since  $(I - \bar{z}B^*)$  is invertible,  $h(z)$  is never zero. Finally a simple geometric series argument shows that  $(T - z)^* h(z) = T^* h_0 = 0$ . Also if  $g \in \mathcal{H}$ , then  $\langle g, h(z) \rangle = \langle g, (I - \bar{z}B^*)^{-1} h_0 \rangle = \langle (I - zB)^{-1} g, h_0 \rangle$  and this is analytic on  $\Delta$  since it has a power series expansion. ■

*Proof of Theorem 2.7.* Let  $\{\lambda_n\}$  be a dense sequence in  $\sigma(T) - \sigma_{ap}(T)$  and let  $\{v_{nk}\}$  be a dense set of nonzero vectors in  $\ker(T - \lambda_n)^*$ . A simple application of Proposition 2.9 shows that  $\{\ker(T - \lambda_n)^*: n \geq 1\}$  has dense linear span in  $\mathcal{H}$ .

Next, Proposition 2.9 implies that there exists disks  $\Delta_{nk}$  centered at  $\lambda_n$  and coanalytic maps  $h_{nk}: \Delta_{nk} \rightarrow \mathcal{H}$  such that  $h_{nk}(\lambda_n) = v_{nk}$  and  $h_{nk}(z) \in \ker(T - z)^* - \{0\}$  for all  $z \in \Delta_{nk}$ . By shrinking the radius slightly of each  $\Delta_{nk}$  we may assume that  $h_{nk}$  is a bounded coanalytic function on  $\Delta_{nk}$ .

Now, define a map  $A: \mathcal{H} \rightarrow (\bigoplus_{nk} H^2(\Delta_{nk}))$  as follows:

$$(Ag)_{nk}(z) = c_{nk} \frac{1}{2^{nk}} \langle g, h_{nk}(z) \rangle$$

where  $c_{nk} = \|h_{nk}\|_\infty^{-1}$ . Notice that  $(Ag)_{nk}$  is a bounded analytic function on  $\Delta_{nk}$ , hence belongs to  $H^2(\Delta_{nk})$ . Furthermore,

$$\|Ag\| = \sum_{n,k} \|(Ag)_{nk}\|_2^2 \leq \sum_{n,k} \|(Ag)_{nk}\|_\infty^2 \leq \|g\| \sum_{n,k} \frac{1}{2^{nk}} \leq \|g\|.$$

Thus  $Ag$  belongs to  $\bigoplus_{nk} H^2(\Delta_{nk})$ . So  $A$  is a bounded linear operator from  $\mathcal{H}$  into  $\bigoplus_{nk} H^2(\Delta_{nk})$ .

We need to show that  $A$  is one-to-one and intertwines  $T$  and  $\bigoplus_{nk} M_z$ .

If  $Ag = 0$  then  $(Ag)_{nk} = 0$  for all  $n, k$ . Thus  $g \perp \{ker(T - \lambda_n)^*: n \geq 1\}$ . But as noticed above this set has dense linear span in  $\mathcal{H}$ . Thus  $g = 0$  and  $A$  is one-to-one.

For the intertwining, if  $g \in \mathcal{H}$ , then we need to show that  $(ATg)_{nk}(z) = z(Ag)_{nk}(z)$  for all  $z \in \Delta_{nk}$ . But if  $z \in \Delta_{nk}$ , then

$$\begin{aligned} (ATg)_{nk}(z) &= c_{nk} \frac{1}{2^{nk}} \langle Tg, h_{nk}(z) \rangle = c_{nk} \frac{1}{2^{nk}} \langle g, T^* h_{nk}(z) \rangle \\ &= c_{nk} \frac{1}{2^{nk}} \langle g, \bar{z} h_{nk}(z) \rangle = c_{nk} \frac{1}{2^{nk}} z \langle g, h_{nk}(z) \rangle = z(Ag)_{nk}(z). \end{aligned}$$

Thus we have that  $AT = (\bigoplus_{nk} M_z) A$ . ■

To illustrate the advantages of the above approach consider the following immediate corollary to Theorem 2.7.

**COROLLARY 2.10.** *If  $T_n \in \mathcal{B}(\mathcal{H}_n)$  is a bounded sequence of operators such that for each  $n$ ,  $\{ker(T_n - \lambda)^*: \lambda \in \sigma(T_n) - \sigma_{ap}(T_n)\}$  has dense linear span, then  $\bigoplus_n T_n^*$  has a dense set of cyclic vectors.*

*Proof.* Theorem 2.7 says that for each operator  $T_n$  there exists disks  $\{\Delta_{nk}: k \geq 0\}$  and a one-to-one linear map  $A_n: \mathcal{H}_n \rightarrow (\bigoplus_k H^2(\Delta_{nk}))$  such that  $A_n T_n = (\bigoplus_k M_z) A_n$ .

Thus, if we set  $A = \bigoplus_n A_n$ , then  $A: \bigoplus_n \mathcal{H}_n \rightarrow (\bigoplus_n (\bigoplus_k H^2(\Delta_{nk})))$  is a one-to-one bounded linear operator that intertwines  $\bigoplus_n T_n$  and  $\bigoplus_{nk} M_z$ . Now an application of Proposition 2.8 completes the proof. ■

### 3. SUBNORMAL OPERATORS

In this section we shall prove that every pure subnormal operator has a cyclic adjoint. An application involving the triangularizability of subnormal operators is also given.

We begin by proving that a subnormal operator  $S$  has a cyclic adjoint if and only if  $S \succ N$  for some cyclic normal operator  $N$ . In doing so we introduce a new form of cyclicity. For general operators this concept lies between the properties of having a cyclic adjoint and being  $*$ -cyclic.

We then proceed to show that every pure subnormal operator  $S$  satisfies the condition that  $S \succ N$  for some cyclic normal operator  $N$ .



If  $S$  is a subnormal operator on a separable Hilbert space  $\mathcal{H}$  and  $N$  is the minimal normal extension of  $S$  on  $\mathcal{H}$ , then let  $P$  denote the projection of  $\mathcal{H}$  onto  $\mathcal{H}$  and  $svsmS$  denote the scalar valued spectral measure of  $N$ . If  $f \in L^\infty(\mu)$ ,  $\mu = svsmS$ , then the Toeplitz operator with symbol  $f$ , denoted by  $S_f$ , is the compression of  $f(N)$  to  $\mathcal{H}$ . Thus,  $S_f = Pf(N)|_{\mathcal{H}}$ .

If  $T$  is any operator, then we shall say that  $T$  is *strongly \*-cyclic* if there exists a (strong \*-cyclic) vector  $v$  such that  $\{T^{*n} T^k v; n, k \geq 0\}$  has dense linear span.

Notice that if  $S$  is a subnormal operator, then  $S^{*n} S^k = S_{\bar{z}^n z^k}$ . Thus it follows that a subnormal operator  $S$  is strongly \*-cyclic if and only if there exists a vector  $v$  such that  $\{S_f v; f \in L^\infty(\mu)\}$  is dense; where  $\mu = svsmS$ .

Notice that in the first result of this section we do not require the subnormal operator to be pure. Whenever purity is needed it will be explicitly stated.

If  $\mu$  is a measure on  $\mathbb{C}$ , then  $N_\mu$  shall denote multiplication by  $z$  on  $L^2(\mu)$ .

**THEOREM 3.1.** *If  $S$  is a subnormal operator, then the following are equivalent:*

1.  $S$  has a cyclic adjoint;
2.  $S$  is strongly \*-cyclic;
3.  $S \succ N_\mu$  where  $\mu$  is a scalar valued spectral measure for  $S$ ;
4.  $S \succ N$  for some cyclic normal operator  $N$ .

*Proof.* In view of Proposition 2.1 it suffices to show that (2) implies (3). So, suppose  $S$  acts on  $\mathcal{H}$ ,  $N$  is the minimal normal extension of  $S$  and  $v \in \mathcal{H}$  is a strong \*-cyclic vector. So  $\{S^{*n} S^k v; n, k \geq 0\}$  has dense linear span.

Notice that  $S^{*n} S^k = S_p$  where  $S_p$  is the Toeplitz operator with symbol  $p(z) = \bar{z}^n z^k$ . Now let  $\mu$  be a scalar valued spectral measure for  $S$ . Consider the map  $T: L^\infty(\mu) \rightarrow \mathcal{H}$  given by  $Tf = S_f v$  where  $S_f$  is the Toeplitz operator with symbol  $f$  (that is the compression of  $f(N)$  to the subspace  $\mathcal{H}$ ).

We want to choose  $v \ll \mu$  such that  $T$  is bounded on  $L^2(\mu)$ . Consider the normal extension  $N$  of  $S$ . Since the map from  $L^\infty(\mu) \rightarrow \mathbb{C}$  given by  $f \mapsto \langle f(N) v, v \rangle$  is a positive weak \*-continuous linear functional, there exists a positive measure  $\nu \ll \mu$  such that  $\langle f(N) v, v \rangle = \int f d\nu$  for all  $f \in L^\infty(\mu)$ .

Let  $P$  be the projection onto  $\mathcal{H}$ . Thus, for  $f \in L^\infty(\mu)$ , we have,

$$\|Tf\|^2 = \|S_f v\|^2 = \|Pf(N) v\|^2 \leq \|f(N) v\|^2 = \int |f|^2 d\nu.$$

Thus  $T$  extends to a bounded linear operator (still denoted by  $T$ ) from  $L^2(\nu) \rightarrow \mathcal{H}$ . Also,  $T: L^2(\nu) \rightarrow \mathcal{H}$  still has dense range; since by assumption  $T(L^\infty(\nu))$  is dense in  $\mathcal{H}$ , and the above inequality implies that  $Tf$  only depends on the values of  $f$   $\nu$ -a.e., thus  $T(L^\infty(\nu)) = T(L^\infty(\mu))$ . Hence  $T(L^\infty(\nu))$  is dense in  $\mathcal{H}$ . Also notice that for  $f \in L^\infty(\mu)$  we have

$$T(\bar{z}f) = S_{\bar{z}f}v = S^* S_f v = S^* T(f).$$

Thus we have  $TN_v^* = S^*T$  on  $L^2(\nu)$ . Hence  $T^*S = N_v T^*$  and since  $T$  has dense range,  $T^*$  is one-to-one. Thus  $S \succ N_v$ . Furthermore if  $N_1$  and  $N_2$  are cyclic normal operators, then  $N_1 \succ N_2$  if and only if  $svsm N_1 \ll svsm N_2$ . Thus we have that  $S \succ N_v \succ N_\mu$ . ■

The next result follows immediately from the previous Theorem and Corollary 2.4.

**COROLLARY 3.2.** *If  $S$  is a subnormal operator with a cyclic adjoint, then  $S^*$  has a dense set of cyclic vectors.*

A more general result concerning the denseness of cyclic vectors for operators was proven in Ansari [1] and Herrero [14], however the above result follows nicely and easily from Corollary 2.4. Also one may easily check that the set of cyclic vectors for any operator forms a  $G_\delta$  set (see the remarks after Theorem 4.5).

In 1955 Bram proved that if  $S = M_z$  on any closed  $z$  invariant subspace  $\mathcal{H} \subseteq L^2(\mu)$ , then  $S$  has a cyclic adjoint. We now prove that the direct sum of a bounded sequence of such operators also has a cyclic adjoint.

**THEOREM 3.3.** *If  $S_n = M_z$  on  $\mathcal{H}_n \subseteq L^2(\mu_n)$  is a bounded sequence of pure subnormal operators, then  $(\bigoplus_n S_n)$  has a cyclic adjoint.*

We begin with some preliminary results. The first result is well known and follows easily from Chaumat's Lemma (see Conway [8], p. 246).

**PROPOSITION 3.4.** *If  $S = M_z$  on  $\mathcal{H} \subseteq L^2(\mu)$ , where  $\mathcal{H}$  is a closed  $z$  invariant subspace of  $L^2(\mu)$ , then  $S$  is pure if and only if there exists a function  $\phi \in L^2(\mu)$  such that  $\phi \perp \mathcal{H}$  and  $|\phi| > 0$   $\mu$ -a.e..*

The next result is fundamental in what follows. We shall say that a closed  $z$  invariant subspace  $\mathcal{H} \subseteq L^2(\mu)$  is *pure* if the operator of multiplication by  $z$  on  $\mathcal{H}$  is a pure operator (that is, if  $\mathcal{H}$  contains no  $L^2$  summand). Notice that in the previous result and the next result, it is not necessary for  $(M_z, L^2(\mu))$  to be the minimal normal extension of  $(M_z, \mathcal{H})$ .

**PROPOSITION 3.5.** *If  $\mathcal{H} \subseteq L^2(\mu)$  is a pure subspace, then there exists a function  $\phi \in L^\infty(\mu)$  such that:*

1.  $|\phi| > 0$   $\mu$ -a.e.;
2. if  $f \in L^2(\mu)$  and  $\phi f \in \mathcal{H}$ , then  $f = 0$   $\mu$ -a.e..

*Proof.* We shall find a function  $\phi \in L^\infty(\mu)$  such that  $|\phi| > 0$   $\mu$ -a.e. and such that  $\bar{\phi} \mathcal{H}^\perp$  is dense in  $L^2(\mu)$ .

Let's show that this will indeed finish the proof. That is, we claim that  $\phi$  also has the property that  $\phi L^2(\mu) \cap \mathcal{H} = (0)$ . To see this, suppose  $f \in L^2(\mu)$  and  $\phi f \in \mathcal{H}$ . It follows that  $\phi f \perp \mathcal{H}^\perp$  or equivalently that  $f \perp \bar{\phi} \mathcal{H}^\perp$ . However,  $\bar{\phi} \mathcal{H}^\perp$  is dense in  $L^2(\mu)$ , thus  $f = 0$   $\mu$ -a.e.. Hence  $\phi$  satisfies property (2).

Let's construct  $\phi$  now. Since  $\mathcal{H}$  is a pure subspace of  $L^2(\mu)$  there exists a function  $g \in \mathcal{H}^\perp$  such that  $|g| > 0$   $\mu$ -a.e.. Since  $\mathcal{H}$  is  $z$  invariant,  $\mathcal{H}^\perp$  is  $\bar{z}$  invariant. Thus  $p(\bar{z})g \in \mathcal{H}^\perp$  for all polynomials  $p$ . By Theorem 2.3  $M_{\bar{z}}$  on  $L^2(|g|^2 d\mu)$  is cyclic and has a bounded cyclic vector, call it  $\bar{\phi}$ .

Thus,  $\{p(\bar{z})\bar{\phi} : p \text{ is a polynomial}\}$  is dense in  $L^2(|g|^2 d\mu)$ . However this happens if and only if  $\{p(\bar{z})g\bar{\phi} : p \text{ is a polynomial}\}$  is dense in  $L^2(\mu)$ . But  $\bar{\phi} \mathcal{H}^\perp \supseteq \{p(\bar{z})g\bar{\phi} : p \text{ is a polynomial}\}$ , thus  $\bar{\phi} \mathcal{H}^\perp$  is dense in  $L^2(\mu)$ ; as required.

To show that  $|\phi| > 0$   $\mu$ -a.e. simply notice that since  $\bar{\phi}$  is a cyclic vector for  $M_{\bar{z}}$  on  $L^2(|g|^2 d\mu)$ , one easily sees that  $|\phi| > 0$  a.e. with respect to  $|g|^2 d\mu$ . However,  $g$  was chosen so that  $|g| > 0$   $\mu$ -a.e.. Thus  $|\phi| > 0$   $\mu$ -a.e.. ■

Let's see a simple application of this result.

**EXAMPLE 3.6.** *If  $S = M_z$  on  $\mathcal{H} \subseteq L^2(\mu)$  is a pure subnormal operator, then  $S \oplus S$  has a cyclic adjoint.*

*Proof.* By Proposition 3.5 there exists a function  $\phi \in L^\infty(\mu)$  such that  $\phi L^2(\mu) \cap \mathcal{H} = (0)$  and  $|\phi| > 0$   $\mu$ -a.e.. Define  $A: \mathcal{H} \oplus \mathcal{H} \rightarrow L^2(\mu)$  by  $A(f, g) = f + \phi g$ . It follows that  $A$  is one-to-one and intertwines  $S \oplus S$  and  $N_\mu (= M_z \text{ on } L^2(\mu))$ . Thus,  $S \oplus S$  has a cyclic adjoint. ■

We now deal with a special case of Theorem 3.3, namely when all the measures are equal. The general case will easily be reduced to this one. In this case the sequence of operators must be bounded. The proof of this special case is a more elaborate version of the previous example.

Recall that  $N_\mu$  denotes the normal operator of  $M_z$  on  $L^2(\mu)$ .

**PROPOSITION 3.7.** *If  $S_n = M_z$  on  $\mathcal{H}_n \subseteq L^2(\mu)$  is a sequence of pure subnormal operators, then  $(\bigoplus_n S_n)$  has a cyclic adjoint.*

*Proof.* We shall construct a sequence of functions  $\{\psi_n\} \subseteq L^\infty(\mu)$  such that  $\|\psi_n\|_\infty \leq 1$  and define a map  $A: \bigoplus_n \mathcal{H}_n \rightarrow L^2(\mu)$  by  $A(\{f_n\}) = \sum_n (1/2^n) \psi_n f_n$ . Clearly,  $A$  intertwines  $(\bigoplus_n S_n)$  and  $N_\mu$ . We must choose the functions  $\{\psi_n\}$  such that  $A$  is one-to-one.

First we define another sequence of functions  $\{\phi_n\}$  as follows. Let  $\phi_1 = 1$ . Now, by Proposition 3.5 for each  $n \geq 1$ , choose  $\phi_{n+1} \in L^\infty(\mu)$  with  $\|\phi_{n+1}\|_\infty = 1$ ,  $|\phi_{n+1}| > 0$   $\mu$ -a.e., and such that  $\phi_{n+1} L^2(\mu) \cap \mathcal{H}_n = 0$ .

Next define  $\psi_n = \prod_{k=1}^n \phi_k$ . So,  $\|\psi_n\|_\infty \leq 1$  and  $|\psi_n| > 0$   $\mu$ -a.e. Now, with this sequence the functions, define the operator  $A$  as above.

As noticed above, it suffices to prove that  $A$  is one-to-one. So, suppose  $f = \{f_n\}$  is a non-zero element of  $\bigoplus_n \mathcal{H}_n$  and that  $Af = 0$ . Let  $k$  be the smallest integer such that  $f_k$  is not identically zero.

So we have the following:

$$\begin{aligned} Af &= \left( \frac{1}{2^k} \psi_k f_k + \frac{1}{2^{k+1}} \psi_{k+1} f_{k+1} + \frac{1}{2^{k+2}} \psi_{k+2} f_{k+2} + \cdots \right) = 0 \\ \psi_k \left( \frac{1}{2^k} f_k + \frac{1}{2^{k+1}} \frac{\psi_{k+1}}{\psi_k} f_{k+1} + \frac{1}{2^{k+2}} \frac{\psi_{k+2}}{\psi_k} f_{k+2} + \cdots \right) &= 0 \\ \left( \frac{1}{2^k} f_k + \frac{1}{2^{k+1}} \frac{\psi_{k+1}}{\psi_k} f_{k+1} + \frac{1}{2^{k+2}} \frac{\psi_{k+2}}{\psi_k} f_{k+2} + \cdots \right) &= 0 \\ \left( \frac{1}{2^k} f_k + \frac{1}{2^{k+1}} \phi_{k+1} f_{k+1} + \frac{1}{2^{k+2}} \phi_{k+1} \phi_{k+2} f_{k+2} + \cdots \right) &= 0 \\ \phi_{k+1} \left( \frac{1}{2^{k+1}} f_{k+1} + \frac{1}{2^{k+2}} \phi_{k+2} f_{k+2} + \cdots \right) &= -\frac{1}{2^k} f_k \in \mathcal{H}_k \end{aligned}$$

But,  $\phi_{k+1}$  was chosen so that  $\phi_{k+1} L^2(\mu) \cap \mathcal{H}_k = (0)$ . Further, the last equation above is of the form  $\phi_{k+1} g = -(1/2^k) f_k \in \mathcal{H}_k$  (where  $g$  is the expression in parenthesis). Thus  $f_k \in \phi_{k+1} L^2(\mu) \cap \mathcal{H}_k$  hence  $f_k = 0$   $\mu$ -a.e.. However this contradicts the choice of the index  $k$ .

Thus we must have that  $A$  is one-to-one. ■

*Remark.* Notice that it was not necessary in the previous result that  $N_\mu$  be the minimal normal extension of  $S_n$ , only that  $S_n$  is pure and the operators have a common cyclic normal extension.

**LEMMA 3.8.** *If  $S = M_z$  on  $\mathcal{H} \subseteq L^2(\mu)$  is a pure subnormal operator and  $\mu \ll \nu$ , then there is a closed pure subspace  $\mathcal{M} \subseteq L^2(\nu)$  such that  $S \cong T$  where  $T = M_z$  on  $\mathcal{M}$ .*

*Proof.* Since  $\mu \ll \nu$ , we have that  $d\mu = w d\nu$  for some non-negative function  $w$ . Let  $U: \mathcal{H} \rightarrow L^2(\nu)$  be given by  $Uf = \sqrt{w}f$ . Note that  $U$  is an

isometry. So, let  $\mathcal{M}$  be the range of  $U$ . Clearly,  $\mathcal{M}$  is a closed  $z$  invariant subspace of  $L^2(v)$  and  $U$  implements a unitary equivalence between  $S$  and  $T$  where  $T = M_z$  on  $\mathcal{M}$ . Also one easily sees that  $\mathcal{M}$  is a pure subspace of  $L^2(v)$ . ■

*Proof of Theorem 3.3.* Suppose that  $S_n = M_z$  on  $\mathcal{H}_n \subseteq L^2(\mu_n)$  is a bounded sequence of pure subnormal operators. The strategy of the proof is to find a *common cyclic* normal extension of the operators  $S_n$  and then apply Proposition 3.7.

Let  $\mu = \sum_n (1/2^n) (\mu_n / \|\mu_n\|)$ . Clearly, for each  $n$ ,  $\mu_n \ll \mu$ , thus the Lemma implies that  $S_n$  is unitarily equivalent to an operator of the form  $T_n = M_z$  on  $\mathcal{M}_n \subseteq L^2(\mu)$ . Thus,  $\bigoplus_n S_n \cong \bigoplus_n T_n$ . Furthermore, since the  $T_n$ 's have a common cyclic normal extension we may apply Proposition 3.7 to obtain the desired result. ■

We now want to reduce the general problem to the special case in Theorem 3.3. To do this, for any pure subnormal operator  $S$  on  $\mathcal{H}$  we want to construct a one-to-one bounded linear operator  $A$  intertwining  $S$  and  $(\bigoplus_n S_n)$  where  $S_n$  is a pure subnormal operator of the form  $S_n = M_z$  on  $\mathcal{H}_n \subseteq L^2(\mu_n)$ .

To accomplish this, we need to construct intertwining maps  $A_n: \mathcal{H} \rightarrow L^2(\mu_n)$  whose ranges are contained in pure subspaces of  $L^2(\mu_n)$  and such that the sequence  $\{A_n\}$  separates points in  $\mathcal{H}$ .

Suppose that  $N$  is a normal operator on  $\mathcal{H}$  and  $\mu = svsm N$ . A vector  $v \in \mathcal{H}$  is a separating vector for  $N$  if whenever  $f \in L^\infty(\mu)$  and  $f(N)v = 0$ , then  $f = 0$   $\mu$ -a.e.. In proving the Spectral Theorem for normal operators, one proves that separating vectors always exists (see Conway [7]). However, we shall need the following stronger result (see Conway [8] p. 249).

**PROPOSITION 3.9.** *If  $S$  is a subnormal operator on  $\mathcal{H}$  and  $N$  is the minimal normal extension of  $S$ , then there is a dense set of vectors in  $\mathcal{H}$  that are separating vectors for  $N$ .*

Observe that the separating vectors for  $N$  and  $N^*$  are the same.

**THEOREM 3.10.** *If  $S$  is a pure subnormal operator, then there exists a bounded sequence of pure subnormal operators  $\{S_n\}$  of the form  $S_n = M_z$  on  $\mathcal{H}_n \subseteq L^2(\mu_n)$  such that  $S \succ (\bigoplus_n S_n)$ .*

*Proof.* Suppose that  $S$  acts on  $\mathcal{H}$  and let  $N$  be the minimal normal extension of  $S$  acting on  $\mathcal{H}$ . By applying Proposition 3.9 to the dual operator,  $\text{dual}(S) = N^*|_{\mathcal{H}^\perp}$ , we see that  $N$  has a dense set of separating vectors in  $\mathcal{H}^\perp$ . So let  $\{y_n\}$  be a sequence of separating vectors for  $N$  that

form a dense subset of  $\mathcal{H}^\perp$ . Also, let  $\mathcal{M}_n$  be the reducing subspace for  $N$  generated by  $y_n$  and let  $P_n$  be the orthogonal projection from  $\mathcal{H}$  onto  $\mathcal{M}_n$ . Thus  $P_n$  commutes with  $N$ .

Since  $N|_{\mathcal{M}_n}$  is a  $*$ -cyclic normal operator, it is unitarily equivalent to an operator of the form  $(M_z, L^2(\mu_n))$  for some measure  $\mu_n$  (actually since  $y_n$  is a separating vector  $\mu_n \approx \text{svsm} S$ ). Let  $U_n: \mathcal{M}_n \rightarrow L^2(\mu_n)$  be an isomorphism that intertwines  $N|_{\mathcal{M}_n}$  and  $M_z$ .

Let  $A_n: \mathcal{H} \rightarrow L^2(\mu_n)$  be given by  $A_n = U_n P_n|_{\mathcal{H}}$ . Notice that since  $P_n$  commutes with  $N$ , it follows that  $A_n$  intertwines  $S$  and  $(M_z, L^2(\mu_n))$ .

We want to show that the range of  $A_n$  is contained in a pure subspace of  $L^2(\mu_n)$ . So, let  $\mathcal{H}_n = \text{cl}[\text{range } A_n]$ . Since  $A_n$  intertwines  $S$  and  $(M_z, L^2(\mu_n))$ , we have that  $\mathcal{H}_n$  is a closed  $z$ -invariant subspace of  $L^2(\mu_n)$ . To see that it is a pure subspace, notice that  $U_n y_n$  is a separating vector for  $(M_z, L^2(\mu_n))$ , hence  $|U_n y_n| > 0$   $\mu_n$ -a.e.. Furthermore, since  $y_n \perp P_n(\mathcal{H})$ , we get that  $U_n y_n \perp \mathcal{H}_n$ ; thus by Proposition 3.4,  $\mathcal{H}_n$  is a pure subspace of  $L^2(\mu_n)$ .

Now, let  $A: \mathcal{H} \rightarrow (\bigoplus_n \mathcal{H}_n)$  be given by  $A = \bigoplus_n A_n$ . Clearly,  $A$  is a bounded linear operator that intertwines  $S$  and  $(\bigoplus_n S_n)$  where  $S_n = M_z$  on  $\mathcal{H}_n \subseteq L^2(\mu_n)$ . We need to establish that  $A$  is one-to-one.

Suppose  $x \in \mathcal{H}$  and  $Ax = 0$ . Thus  $A_n x = 0$  for all  $n$  or equivalently,  $P_n x = 0$  for all  $n$ . Thus,  $x \perp \mathcal{M}_n$  for all  $n$ . Thus,  $\langle x, N^{*i} N^j y_n \rangle = 0$  for all  $i, j$  and  $n$ . Since the sequence  $\{y_n\}$  is dense in  $\mathcal{H}^\perp$ , it follows that  $x \perp \mathcal{M}$  where  $\mathcal{M}$  is the smallest reducing subspace for  $N$  containing  $\mathcal{H}^\perp$ . However, since  $S$  is pure, it follows that  $\mathcal{M} = \mathcal{H}$  (otherwise  $\mathcal{M}^\perp$  is a reducing subspace for  $N$  contained in  $\mathcal{H}$ ). Hence  $x = 0$  and  $A$  is one-to-one. Thus  $S \succ (\bigoplus_n S_n)$ . ■

Now by Theorem 3.3, we have the desired result.

**COROLLARY 3.11.** *Every pure subnormal operator has a cyclic adjoint.*

Let's say that a matrix  $(a_{ij})$  is *almost lower triangular* if there are no non-zero entries above the super diagonal; that is, if  $a_{ij} = 0$  whenever  $j > i + 1$ . Similarly a matrix is called *almost upper triangular* if its transpose is almost lower triangular.

**COROLLARY 3.12.** *Every pure subnormal operator has a matrix representation that is almost lower triangular.*

*Proof.* Let  $S$  be a pure subnormal operator and  $v$  a cyclic vector for  $S^*$ . The sequence  $\{v, S^*v, S^{*2}v, \dots\}$  is linearly independent and has dense linear span. If  $\{e_n\}$  is the orthonormal basis obtained from the Gram-Schmidt process applied to  $\{v, S^*v, S^{*2}v, \dots\}$ , then the matrix representation of  $S^*$  in this basis is almost upper triangular. Thus, with respect to the basis

$\{e_n\}$ ,  $S$  has an almost lower triangular matrix representation. The idea of this proof is due to Halmos [12]. ■

Notice that if an operator  $S$  has a lower triangular matrix representation, then  $S^*$  has an upper triangular matrix representation, hence  $S^*$  has finite dimensional invariant subspaces; thus  $S^*$  has eigenvectors. So if  $S$  is any pure subnormal operator such that  $S^*$  has no eigenvectors, then  $S$  does not have a lower triangular matrix representation. The dual of the Bergman operator,  $S = M_{\bar{z}}$  on  $L_a^2(\mathbb{D})^\perp$ , is one such operator. Thus, Corollary 3.12 is the best possible result.

#### 4. ADDITIONAL RESULTS

In this section we present a few additional results. In particular, we shall consider subnormal operators that are not pure and determine which ones have cyclic adjoints. We shall show that a subnormal operator has a cyclic adjoint precisely when its normal part is cyclic. Or equivalently, a subnormal operator has a cyclic adjoint if and only if it is  $*$ -cyclic. In this sense, subnormal operators behave exactly as normal operators.

We also characterize the strong  $*$ -cyclic vectors for a subnormal operator in terms of intertwining maps and prove that every pure hyponormal operator is  $*$ -cyclic.

Recall that every subnormal operator  $S$  may be written in the form  $S = S_p \oplus N$  where  $S_p$  is a pure subnormal operator and  $N$  is a normal operator.

**THEOREM 4.1.** *If  $S = S_p \oplus N$  is a subnormal operator, with  $S_p$  pure and  $N$  normal, then  $S$  has a cyclic adjoint if and only if  $N$  is cyclic.*

First a few preliminaries. For a subnormal operator  $S$ , let  $svsmS$  denote the scalar valued spectral measure of the minimal normal extension of  $S$ . The following result is an easy corollary of Theorem 3.1.

**PROPOSITION 4.2.** *If  $S$  and  $T$  are subnormal operators with cyclic adjoints and  $svsmS$  is mutually singular with respect to  $svsmT$ , then  $S \oplus T$  has a cyclic adjoint.*

*Proof.* Suppose that  $S$  and  $T$  act on  $\mathcal{H}$  and  $\mathcal{K}$  respectively. Let  $\mu = svsmS$  and  $\nu = svsmT$ . Since  $S$  and  $T$  have cyclic adjoints, by Theorem 3.1 there exists one-to-one maps  $A: \mathcal{H} \rightarrow L^2(\mu)$  and  $B: \mathcal{K} \rightarrow L^2(\nu)$  such that  $AS = N_\mu A$  and  $BT = N_\nu B$ .

Now,  $(A \oplus B): \mathcal{H} \oplus \mathcal{H} \rightarrow L^2(\mu) \oplus L^2(\nu)$  is a one-to-one map that intertwines  $S \oplus T$  and  $N_\mu \oplus N_\nu$ . However, since  $\mu \perp \nu$ ,  $N_\mu \oplus N_\nu \cong N_{\mu+\nu}$ . Thus,  $(S \oplus T) \succ N_{\mu+\nu}$ , so  $S \oplus T$  has a cyclic adjoint. ■

**PROPOSITION 4.3.** *If  $S$  is a subnormal operator, then the following are equivalent:*

1.  $S \oplus N$  has a cyclic adjoint for every cyclic normal operator  $N$ .
2.  $S \oplus N_\mu$  has a cyclic adjoint where  $\mu = svsmS$ .

*Proof.* Clearly (1) implies (2). We shall prove that (2) implies (1).

Let  $\mu = svsmS$  and suppose that  $S \oplus N_\mu$  has a cyclic adjoint. Let  $N$  be any cyclic normal operator. So  $N \cong N_\nu$  for some measure  $\nu$ . Write  $\nu = \nu_a + \nu_s$  where  $\nu_a \ll \mu$  and  $\nu_s \perp \mu$ . Thus,  $N \cong N_\nu \cong N_{\nu_a} \oplus N_{\nu_s}$ .

Since,  $\nu_a \ll \mu$ , we have  $N_{\nu_a} \succ N_\mu$  thus  $S \oplus N_{\nu_a} \succ S \oplus N_\mu$ . Since we are assuming that  $S \oplus N_\mu$  has a cyclic adjoint, we get that  $S \oplus N_{\nu_a}$  also has a cyclic adjoint.

Now since  $S \oplus N_{\nu_a}$  has a cyclic adjoint, and  $N_{\nu_s}$  has a cyclic adjoint, and these operators have singular spectral measures, Proposition 4.2 implies that the direct sum has a cyclic adjoint. Hence  $S \oplus N$  has a cyclic adjoint. ■

**PROPOSITION 4.4.** *If  $S_k = M_z$  on  $\mathcal{H}_k \subseteq L^2(\mu_k)$  is a pure subnormal operator for  $1 \leq k \leq n$ , then  $S_1 \oplus S_2 \oplus \cdots \oplus S_n \oplus N$  has a cyclic adjoint for any cyclic normal operator  $N$ .*

*Proof.* It suffices to show that  $S_1 \oplus S_2 \oplus \cdots \oplus S_n \oplus N_\mu$  has a cyclic adjoint where  $\mu = svsm(S_1 \oplus S_2 \oplus \cdots \oplus S_n)$ . Since  $\mu_k \ll \mu$ , Proposition 3.8 says that we may assume that  $\mathcal{H}_k \subseteq L^2(\mu)$  for all  $k$ .

Now, Proposition 3.5 gives a bounded function  $\phi_k$  such that  $|\phi_k| > 0$   $\mu$ -a.e. and  $\phi_k L^2(\mu) \cap \mathcal{H}_k = (0)$ . Also, define  $\psi_k = \phi_1 \phi_2 \cdots \phi_k$  for  $1 \leq k \leq n$ .

Now define a map  $A: \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n \oplus L^2(\mu) \rightarrow L^2(\mu)$  by

$$A(f_1, \dots, f_n, g) = f_1 + \psi_1 f_2 + \cdots + \psi_{n-1} f_n + \psi_n g$$

Clearly  $A$  intertwines  $S_1 \oplus S_2 \oplus \cdots \oplus S_n \oplus N_\mu$  and  $N_\mu$ . Also one may check that  $A$  is one-to-one as in the proof of Proposition 3.7. So,  $S_1 \oplus S_2 \oplus \cdots \oplus S_n \oplus N_\mu$  has a cyclic adjoint. ■

We shall also need the following result that was essentially proven by Herrero and Wogen [15] (see the proof of Theorem 2.1). However, we shall sketch another proof. The author would like to thank Hector Salas for pointing out this alternative approach.



**THEOREM 4.5.** *If  $\{T_n\}$  is a bounded sequence of operators such that  $T_1 \oplus \cdots \oplus T_n$  has a dense set of cyclic vectors for each  $n \geq 1$ , then  $\bigoplus_{n=1}^{\infty} T_n$  also has a dense set of cyclic vectors.*

The proof of this Theorem requires a basic lemma. For an operator  $T$ , let  $\mathcal{A}$  denote the algebra of all polynomials in  $T$ . Notice that if  $\{U_n\}$  is a basis for the topology, then  $\bigcap_n \bigcup_{A \in \mathcal{A}} A^{-1}(U_n)$  is the set of cyclic vectors for  $T$ , where  $A^{-1}(V)$  denotes the inverse image of the set  $V$ .

The next result follows almost from the definitions (and the Baire Category Theorem), its proof is left to the reader.

**LEMMA 4.6.** *If  $T$  is a bounded linear operator on  $\mathcal{H}$  and  $\mathcal{A}$  is the algebra of all polynomials in  $T$ , then the following are equivalent:*

1.  $T$  has a dense set of cyclic vectors;
2. For any pair of nonempty open sets  $U, V$  there exists an  $A \in \mathcal{A}$  such that  $A(U) \cap V \neq \emptyset$ ;

*Proof of Theorem 4.5.* Assume that  $T_n$  acts on the space  $\mathcal{H}_n$ . Our assumption is that  $T_1 \oplus \cdots \oplus T_n$  has a dense set of cyclic vectors in  $\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$ . Thus, condition (2) above holds.

We shall check that (2) holds for  $T = \bigoplus_{n=1}^{\infty} T_n$ . A basis for the topology of  $\mathcal{H} = (\bigoplus_{n=1}^{\infty} \mathcal{H}_n)$  consists of open balls and it suffices to check condition (2) for basis elements.

So, fix two open balls  $U = B(\mathbf{x}, \varepsilon_1)$  and  $V = B(\mathbf{y}, \varepsilon_2)$  where  $\mathbf{x} = (x_1, x_2, \dots)$ ,  $\mathbf{y} = (y_1, y_2, \dots)$  and  $\varepsilon_1, \varepsilon_2 > 0$ . For  $n \geq 1$  let  $\mathbf{x}_n = (x_1, \dots, x_n, 0, \dots)$  and  $\mathbf{y}_n = (y_1, \dots, y_n, 0, \dots)$ . Also for convenience, let  $\hat{\mathbf{x}}_n = (x_1, \dots, x_n)$  and  $\hat{\mathbf{y}}_n = (y_1, \dots, y_n)$ .

Now choose  $n$  large enough such that  $\mathbf{x}_n \in B(\mathbf{x}, \varepsilon_1/2)$  and  $\mathbf{y}_n \in B(\mathbf{y}, \varepsilon_2/2)$ . Since,  $T_1 \oplus \cdots \oplus T_n$  satisfies condition (2), there exists a vector  $\mathbf{t}_n = (t_1, \dots, t_n) \in B(\hat{\mathbf{x}}_n, \varepsilon_1/2)$  and a polynomial  $p$  such that  $(p(T_1)t_1, \dots, p(T_n)t_n) \in B(\hat{\mathbf{y}}_n, \varepsilon_2/2)$ .

Thus,  $\mathbf{t} = (t_1, \dots, t_n, 0, \dots) \in U$  and  $p(T)\mathbf{t} \in V$ . So,  $T$  has a dense set of cyclic vectors. ■

We are now ready to prove Theorem 4.1.

*Proof of Theorem 4.1.* Suppose  $S = S_p \oplus N$  where  $S_p$  is a pure subnormal operator and  $N$  is a normal operator. If  $S$  has a cyclic adjoint, then clearly  $N$  must be a cyclic operator. So, suppose that  $N$  is cyclic. By, Theorem 3.10, there exists pure subnormal operators  $S_n$  of the form  $S_n = M_z$  on  $\mathcal{H}_n \subseteq L^2(\mu_n)$  such that  $S = (S_p \oplus N) \succ (\bigoplus_n S_n) \oplus N$ .

Hence it suffices to show that  $(\bigoplus_n S_n) \oplus N$  has a cyclic adjoint. Now, by Proposition 4.4  $N \oplus S_1 \oplus S_2 \oplus \cdots \oplus S_n$  has a cyclic adjoint for all  $n \geq 1$ . Since by Corollary 3.2 the adjoints of each of these finite direct sums has

a dense set of cyclic vectors, we may conclude by Theorem 4.5 that  $(\bigoplus_n S_n) \oplus N$  has a cyclic adjoint. Thus  $S$  also has a cyclic adjoint. ■

We now characterize the strong  $*$ -cyclic vectors for subnormal operators in terms of natural intertwining maps. Even though a subnormal operator  $S$  has a cyclic adjoint precisely when it is strongly  $*$ -cyclic (see Theorem 3.1), the cyclic vectors for  $S^*$  and the strong  $*$ -cyclic vectors differ dramatically.

Let's set some notation. Let  $S$  be a subnormal operator on  $\mathcal{H}$  and let  $N$  be its minimal normal extension on  $\mathcal{K}$ . If  $v \in \mathcal{H}$ , then let  $\mathcal{M}$  be the reducing subspace for  $N$  generated by  $v$ . Also, let  $P_{\mathcal{M}}$  be the projection of  $\mathcal{K}$  onto  $\mathcal{M}$ . Since  $P_{\mathcal{M}}$  commutes with  $N$ , it follows that  $P_{\mathcal{M}}|_{\mathcal{H}}: \mathcal{H} \rightarrow \mathcal{M}$  intertwines  $S$  and  $N|_{\mathcal{M}}$ ; and the latter operator is a cyclic normal operator. Furthermore, this intertwining map  $P_{\mathcal{M}}|_{\mathcal{H}}$  is never zero (provided that  $v \neq 0$ ). For if  $P_{\mathcal{M}}|_{\mathcal{H}} = 0$ , then  $\mathcal{H} \perp \mathcal{M}$ . Thus  $\mathcal{H} \subseteq \mathcal{M}^\perp$ , and this says that  $N$  is not the minimal normal extension of  $S$ . Thus there is an abundance of non-trivial maps intertwining  $S$  and a cyclic normal operator.

**PROPOSITION 4.7.** *If  $v \in \mathcal{H}$ , then  $v$  is a strong  $*$ -cyclic vector for  $S$  if and only if the intertwining map  $P_{\mathcal{M}}|_{\mathcal{H}}: \mathcal{H} \rightarrow \mathcal{M}$  is one-to-one.*

*Proof.* Let  $\mu = svsmS$ . A vector  $v \in \mathcal{H}$  is a strong  $*$ -cyclic vector for  $S$ , if and only if  $\{S_f v: f \in L^\infty(\mu)\}$  is dense in  $\mathcal{H}$ . Since  $S_f = P_{\mathcal{H}} f(N)|_{\mathcal{H}}$  we see that the above set is dense precisely when  $P_{\mathcal{H}}(\mathcal{M})$  is dense in  $\mathcal{H}$ . But  $(P_{\mathcal{H}}|_{\mathcal{M}})^* = (P_{\mathcal{M}}|_{\mathcal{H}})$ . So,  $(P_{\mathcal{H}}|_{\mathcal{M}})$  will have dense range if and only if  $(P_{\mathcal{M}}|_{\mathcal{H}})$  is one-to-one. ■

**COROLLARY 4.8.** *If  $S = M_z$  on  $\mathcal{H} \subseteq L^2(\mu)$ , then a function  $\phi \in \mathcal{H}$  is a strong  $*$ -cyclic vector for  $S$  if and only if no non-zero function in  $\mathcal{H}$  vanishes  $\mu$ -a.e. on  $\{|\phi| > 0\}$ .*

As an example, notice that while the cyclic vectors for the adjoint of the Bergman shift admit no reasonable description, the strong  $*$ -cyclic vectors for any Bergman operator are easily described.

**EXAMPLE 4.9.** If  $G$  is a bounded region and  $S = M_z$  on the Bergman space  $L_a^2(G)$ , then every non-zero function in  $L_a^2(G)$  is a strong  $*$ -cyclic vector for  $S$ .

Another nice example of this difference arises with the unilateral shift. For the unilateral shift  $S$  on  $H^2(\mathbb{D})$ , a function is a cyclic vector for  $S^*$  if and only if it does not have a pseudocontinuation to the exterior of the unit disk (see Douglas, Shapiro, and Shields [10]). However, Corollary 4.8 implies that every non-zero function in  $H^2$  is a strong  $*$ -cyclic vector for  $S$ .

It also follows from Corollary 4.8 that a strong  $*$ -cyclic vector is not necessarily a separating vector for the minimal normal extension of  $S$  (consider a cyclic subnormal operator whose spectral measure has point masses).

We conclude by showing that an obvious necessary condition, for a pure hyponormal operator to have a cyclic adjoint, is satisfied. Namely, we will show that every pure hyponormal operator is  $*$ -cyclic.

**THEOREM 4.10.** *If  $T$  is an operator such that  $W^*(T)$  is properly infinite, then  $T$  is  $*$ -cyclic.*

*Proof.* The proof entails showing that  $W^*(T)$  contains a cyclic operator. Since  $W^*(T)$  is properly infinite it can be written as  $W^*(T) = \mathcal{A} \otimes \mathcal{B}(\mathcal{H})$  where  $\mathcal{A}$  is a properly infinite von Neumann algebra and  $\mathcal{H}$  is a separable infinite dimensional Hilbert space (see [17], p. 48). Thus, elements of  $W^*(T)$  may be considered as (infinite) matrices with entries from  $\mathcal{A}$ . In particular, we may construct a unilateral shift in  $W^*(T)$  by forming a matrix with the identity,  $I$ , along the subdiagonal and zeros elsewhere. Call this shift  $S$ .

It follows that any cyclic vector for  $S^*$  will be a  $*$ -cyclic vector for  $T$ ; and since  $S$  is a pure isometry we know that  $S^*$  is cyclic (represent  $S$  as multiplication by an inner function on  $H^2(\mathbb{D})$  and apply Theorem 2.5). Hence  $T$  is  $*$ -cyclic. ■

In [2] Behncke shows that no finite nonabelian von Neumann algebra is generated by a hyponormal operator. Thus it follows that a pure hyponormal operator must generate a properly infinite von Neumann algebra.

**COROLLARY 4.11.** *A pure hyponormal operator is  $*$ -cyclic.*

In view of the previous Corollary, the following is equivalent to Theorem 4.1.

**COROLLARY 4.12.** *A subnormal operator has a cyclic adjoint if and only if it is  $*$ -cyclic.*

In this sense, subnormal operators behave exactly as normal operators.

**QUESTION 4.13.** *If  $S$  is a pure subnormal operator, then does  $S \succ (M_z, \mathcal{H})$  where  $\mathcal{H}$  is a closed pure  $z$ -invariant subspace of  $L^2(\mu)$ ?*

**QUESTION 4.14.** *Is every pure subnormal operator quasisisimilar to a direct sum of pure subnormal operators each having a cyclic normal extension?*

The previous two questions relate to improvements of Theorem 3.10.

QUESTION 4.15. *If  $S$  is a pure subnormal operator, then is there a common cyclic vector for the adjoints of the pure operators in  $P^\infty(S)$ ?*

The previous question is a natural generalization of Wogen's result in [18], where he gave an affirmative answer to the question for the unilateral shift. Also, Chan [5] and Bourdon and Shapiro [4] have given affirmative answers to the above question for multiplication operators on spaces of analytic functions.

Below are some natural questions aimed at extending some of the results of this paper to hyponormal operators.

QUESTION 4.16. *If  $T$  is a hyponormal operator with a cyclic adjoint, then does  $T \succ N$  for some cyclic normal operator  $N$ ?*

QUESTION 4.17. *If a hyponormal operator is strongly  $*$ -cyclic, then must it have a cyclic adjoint?*

It is known that every pure hyponormal operator with rank one self-commutator is strongly  $*$ -cyclic, see Martin and Putinar [16], page 42.

QUESTION 4.18. *If  $T$  is a pure hyponormal operator, then does  $T \succ N$  for some cyclic normal operator  $N$ ?*

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